Hash functions from abelian threefolds

Leuven Isogeny Days

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The CGL hash function

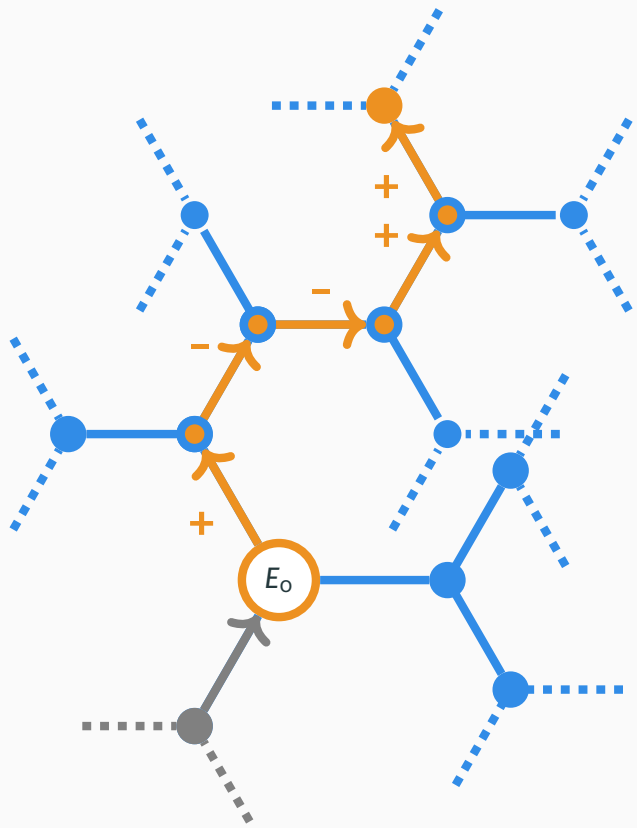
The Charles-Goren-Lauter hash function '08

Setup: supersingular elliptic curve $E_0/\mathbb{F}_{p^2}$
Input: message $(m_1 \dots m_n)$
Output: $H(m) \in \mathbb{F}_{p^2}$

for $i = 1, \dots, n$ **do**
    $E_i = \text{radical2iso}(E_{i-1}, m_i)$
return $j(E_n)$

radical2iso computes a radical 2-isogeny

- cost: essentially one sqrt
- m_i decides the sign of the sqrt



$m = (+ - - + +)$

Security assumptions

Preimage resistance

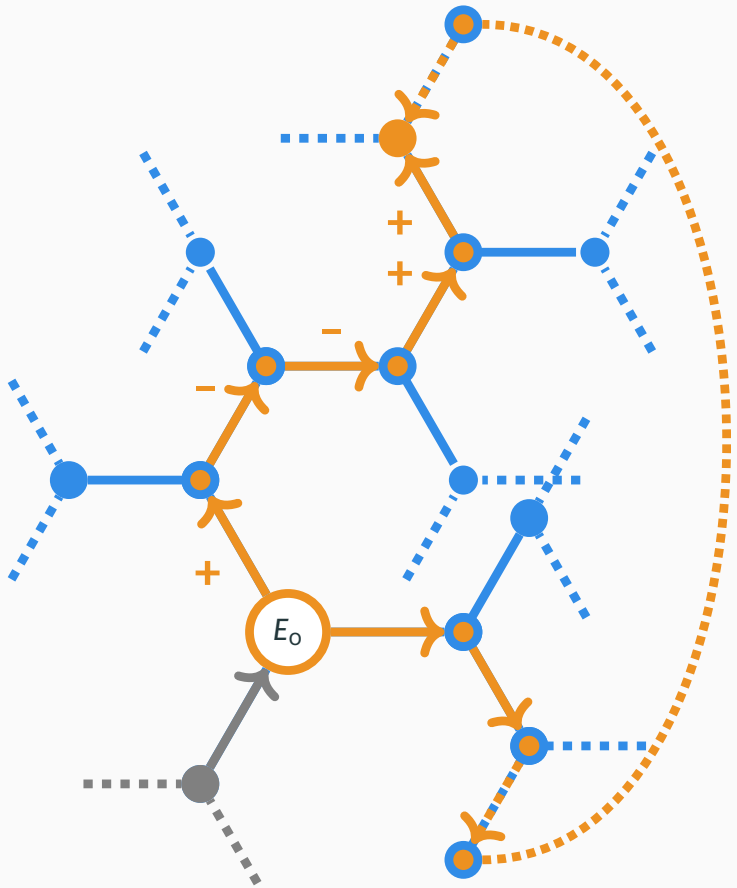
- Given $j \in \mathbb{F}_{p^2}$, it is hard to find m with $H(m) = j$.

$\Rightarrow$ finding a 2^*-isogeny between two supersingular elliptic curves

Collision resistance

- It is hard to find two messages $m \neq m'$ with $H(m) \neq H(m')$.

$\Rightarrow$ finding an endomorphism of degree 2^* of E_0.



CGL in dimension 2

The 2-dimensional setting

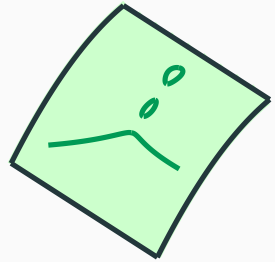
Principally polarized abelian surfaces

Product of elliptic curves,

$E \times E'$

$\approx p^2$

Jacobian of genus-2 curve C,



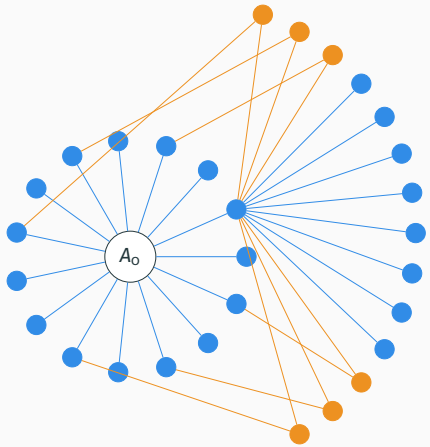
$C : y^2 = f(x)$, $\deg(f) \in \{5, 6\}$.

$\approx p^3$

number of superspecial varieties

(N, N) - Isogenies

- **Kernel**: maximal isotropic subgroup of $A[N]$, isomorphic to $(\mathbb{Z}/N\mathbb{Z})^2$.
- $N = \ell$ prime: $\ell^3 + \ell^2 + \ell + 1$ isotropic (ℓ, ℓ)-groups in total.
- Non-backtracking composition of two (ℓ, ℓ)-isogenies:
  (ℓ^2, ℓ^2)-isogeny , (ℓ^2, ℓ, ℓ)-isogeny



setting for $\ell = 2$

For an (ℓ, ℓ)-isogeny ϕ_1, we are only interested in *good* extensions, i.e. ϕ_2 so that $\phi_2 \circ \phi_1$ is an (ℓ^2, ℓ^2)-isogeny.

$\Rightarrow$ **Multiradical formulas**: explicitly known for $\ell = 2, 3$ (Castryck-Decru '21)

CGL - dim2: Takashima '18, Castryck-Decru-Smith '20

Setup: superspecial abelian surface $A_0/\mathbb{F}_{p^2}$
Input: message $(m_1 \dots m_{3n})$
Output: $H(m) \in \mathbb{F}_{p^2}$

for $i = 1, \dots, n$ **do**
    $A_i = \text{radical2iso}(A_{i-1}, m_{3i-2}, m_{3i-1}, m_{3i})$
return $\text{Igusa}(A_n)$

radical2iso (Richelot)

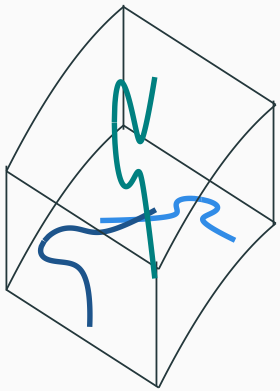
Input: $A = Jac(C)$, $C : y^2 = l_1(x) \cdots l_6(x)$
Output: $A' = Jac(C')$, $C' : y^2 = l'_1(x) \cdots l'_6(x)$.
(1) $g_1 = l_1 \cdot l_4, g_2 = l_2 \cdot l_5, g_3 = l_3 \cdot l_6$
(2) $h_1 = g_2 g'_3 - g_3 g'_2, h_2 = g_3 g'_1 - g_1 g'_3,$
$h_3 = g_1 g'_2 - g_2 g'_1$
(3) $l'_1 \cdot l'_2 = h_1, l'_3 \cdot l'_4 = h_2, l'_5 \cdot l'_6 = h_3$

- cost: essentially 3 sqrts
- $m_{3i-2}, m_{3i-1}, m_{3i}$ decide the signs of the sqrts

The 3-dimensional picture

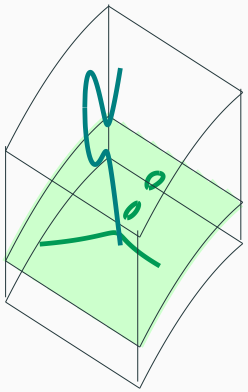
Principally polarized abelian threefolds

reducible threefolds
product of elliptic curve E with abelian surface A



$\approx p^3$

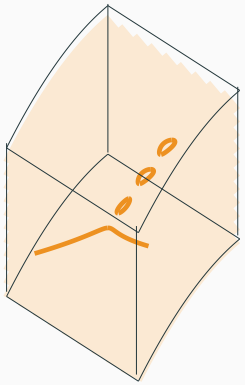
A reducible



$\approx p^4$

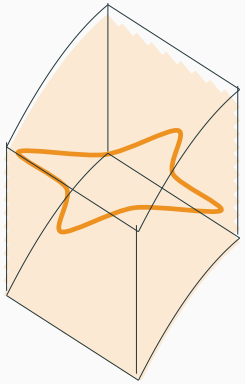
A irreducible

irreducible threefolds
Jacobian of a genus-3 curve C



$\approx p^5$ (?)

C hyperelliptic



$\approx p^6$

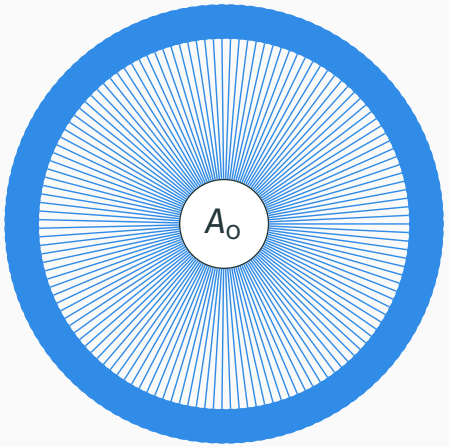
C plane quartic

number of superspecial varieties

Isogenies in dimension 3

(N, N, N)-isogenies

- kernel $G \subset A[N]$ maximal isotropic, and $G \cong \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.
- $N = \ell$ prime: $(\ell^3 + 1)(\ell^2 + 1)(\ell + 1)$ isotropic (ℓ, ℓ, ℓ)-groups of $A[\ell]$ in total.



setting for $\ell = 2$

Computational aspects

- Explicit radical formulas for $\ell = 2$ using theta functions (Ohashi-Onuki-Yoshizumi-Kudo-Nuida '24, this work)

Radical $(2, \ldots, 2)$-isogenies in theta coordinates

$(2,\dots,2)$-isogenies in theta coordinates

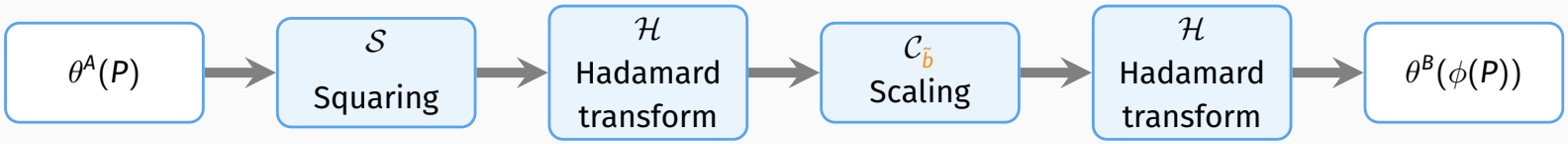
Notation

- A: principally polarized abelian variety of dimension g.
- $\theta^A : A \to \mathbb{P}^{2^g-1}$: a level-2 theta structure.
- $\theta^A(0_A) = (a_0 : \cdots : a_{2^g-1})$: the level-2 theta null point of (A, θ^A).

A maximal 2-isotropic group

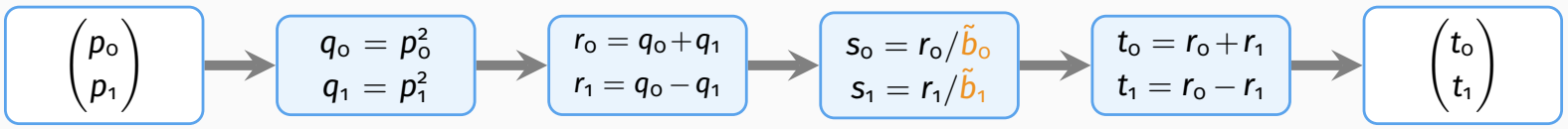
$$G = \langle (a_0 : -a_1 : \cdots : a_{2^g-2} : -a_{2^g-1}), \dots, (a_0 : a_1 : \cdots : -a_{2^g-2} : -a_{2^g-1}) \rangle$$

The $(2,\dots,2)$-isogeny [1] $\phi : A \to B = A/G$



Example: $g = 1$

$\tilde{b}$ dual theta null point of (B, θ^B)



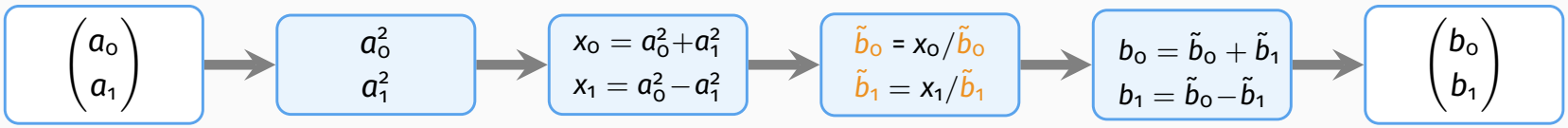
[1] Building block for computing $(2^n, 2^n)$-isogenies (Dartois-Maino-Pope-Robert'24)

Radical formula for $\phi : A \to B$

Goal:
On input $\theta^A(\mathrm{o}_A) = (a_0 : \cdots : a_{2^g-1})$, output $\theta^B(\mathrm{o}_B) = (b_0 : \cdots : b_{2^g-1})$.

Idea: $\mathrm{o}_B = \phi(\mathrm{o}_A)$.

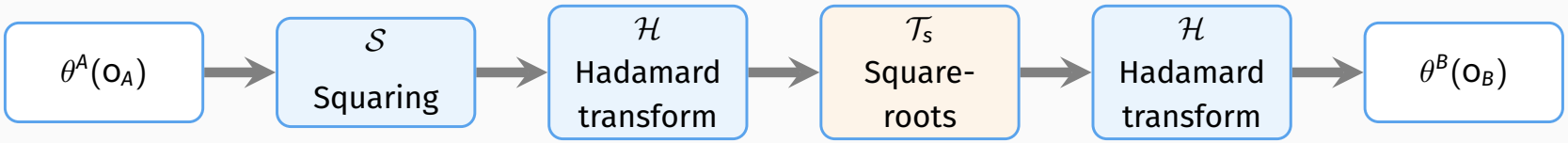
For $g = 1$ $\qquad \tilde{b}_i^2 = x_i$



In general dimension g:

$$\tilde{b}_i^2 = x_i \quad \text{for } i = 0, \ldots, 2^g - 1.$$

Radical isogeny



Compatible square-roots

Essentially: $\mathcal{T}_s : (x_0 : \cdots : x_{2^g-1}) \to (\pm\sqrt{x_0} : \cdots : \pm\sqrt{x_{2^g-1}})$

- Fact: A radical $(2, \ldots, 2)$-isogeny requires $g(g+1)/2$ square-root computations.

⚠ We have $2^g > g(g+1)/2$ coordinates. too many isogenies?

Trick: $(\sqrt{x_0} : \cdots : \sqrt{x_{2^g-1}}) = (x_0 : \sqrt{x_0 x_1} : \cdots : \sqrt{x_0 x_{2^g-1}})$.
$\Rightarrow$ left with $2^g - 1 \geq g(g+1)/2$ square-root computations.

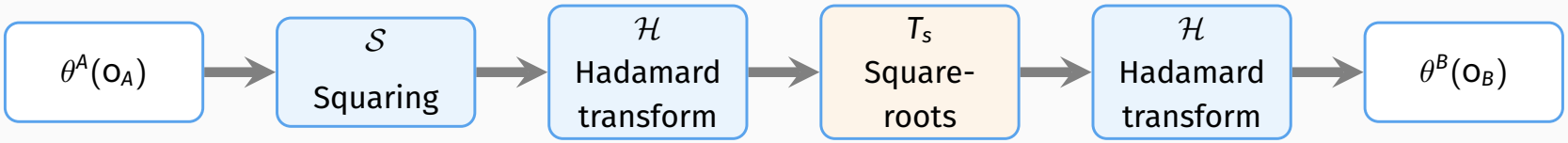
Lemma

Let $I = \{i_1, \ldots, i_{g(g+1)/2}\}$, [a] and assume $x_i \neq 0$ for $i \in I \cap \{0\}$. Then fixing the square-roots $\sqrt{x_0 x_i}$ with $i \in I$, uniquely determines the remaining square-roots.

[a] There are some restrictions on the index set I

Radical isogenies in dimensions 1 and 2

General description



for $g = 1, 2$, we have $2^g - 1 = g(g+1)/2$

dimension 1:

$g(g+1)/2 = 2^g - 1 = 1$

$\mathcal{T}_s : (x_0 : x_1) \mapsto$
$(x_0 : s \cdot \sqrt{x_0 x_1})$

dimension 2:

$g(g+1)/2 = 2^g - 1 = 3$

$\mathcal{T}_s : (x_0 : x_1 : x_2 : x_3) \mapsto$
$(x_0 : s_1 \cdot \sqrt{x_0 x_1} : s_2 \cdot \sqrt{x_0 x_2} : s_3 \cdot \sqrt{x_0 x_3})$

Radical isogenies in dimension 3

$$2^g - 1 = 7 > 6 = g(g+1)/2$$

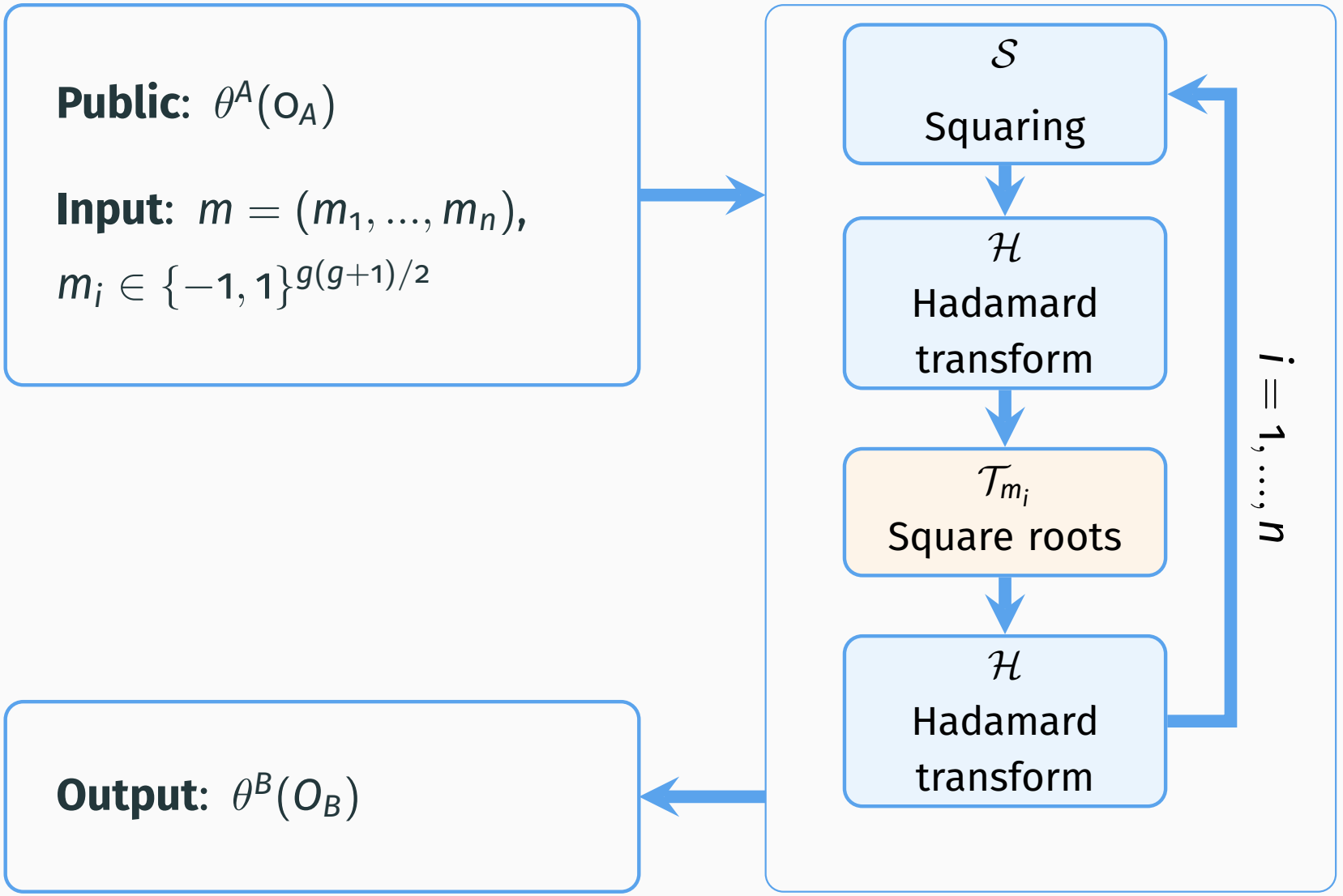
Theorem

$\mathcal{T}_s : (x_0 : \cdots : x_7) \mapsto (x_0 : s_1\sqrt{x_0x_1} : \cdots : s_6\sqrt{x_0x_6} : y_7)$, where
$y_7 = F(s_1\sqrt{x_0x_1}, \ldots, s_6\sqrt{x_0x_6}, x_7)$
for an explicit algebraic expression F.

- **Evaluation of F:** 22 multiplications (in the generic case).
- **Proof idea:** Not all points $(y_0 : \cdots : y_7)$ correspond to a p.p. abelian threefold.
  We use the *Riemann relations* to find a relation $G(y_0, \ldots, y_7) = 0$ among the coordinates of a valid theta null point.

Theta-CGL

Description of our hash function: Theta-CGL$_g$



Why go to dimension $g > 1$?

	$g = 1$	$g = 2$	$g = 3$	$g > 1$
superspecial a.v.s	$\approx p$	$\approx p^3$	$\approx p^6$	$p^{g(g+1)/2}$
solving the isogeny problem	$O(\sqrt{p})$	$O(p)$	$O(p^2)$	$O(p^{g-1})$

Theta-CGL$_g$ with security parameter $\lambda = 128$

A: irreducible superspecial p.p.a.v. over $\mathbb{F}_{p^2}$

	$g = 1$	$g = 2$	$g = 3$	$g > 1$
$\log p$	256	128	64	$\lambda/(g-1)$
computations per bit	$\approx$ 1 sqrt	$\approx$ 1 sqrt	$\approx$ 1 sqrt+	?

Going to higher dimensions allows us to significantly reduce the size of the prime, essentially at no additional cost!

Implementation in Rust

Running times of computing the hash of a 256-bit message in dimensions one, two and three.

	2-radical (μs)	4-radical (μs)	8-radical (μs)
$g = 1$	3153	2037	1737
$g = 2$	989	742	—
$g = 3$	432	—	—

Summary

This presentation

- Explicit formulas for radical $(2, \ldots, 2)$-isogenies in dimensions $g = 1, 2, 3$ using level-2 theta structure.
- Implementation of **Theta-CGL**$_g$ for $g = 1, 2, 3$ showing that dimension 3 is the best setting!

Our (upcoming) paper

- Radical isogeny formulas for 4- and 8-isogenies ($g = 1$)
- Almost radical isogeny formulas for $(4, 4)$-isogenies ($g = 2$)
- Non-generic radical $(2, 2, 2)$-isogeny formulas (e.g. splitting and gluing isogenies).

Thanks for your attention!